9-5: Limits and L'Hopital's Rule

Remember in the limit chapter that, when substituting yields $\frac{0}{0}$, limits are evaluated by factoring and canceling.

EX 1 Find $\lim_{x \to -3} \frac{2x^3 + x^2 - 13x + 6}{x^2 + x - 6}$

$$\begin{aligned}\lim_{x \to -3} \frac{2x^3 + x^2 - 13x + 6}{x^2 + x - 6} &= \lim_{x \to -3} \frac{(x+3)(2x^2 - 5x + 2)}{(x+3)(x-2)} \\ &= \lim_{x \to -3} \frac{(2x^2 - 5x + 2)}{(x-2)} \\ &= \frac{35}{-5} \\ &= -7\end{aligned}$$

Or multiply by conjugates to reveal the factor that led to $\frac{0}{0}$.

EX 2 Find $\lim_{x \to 2} \frac{\sqrt{4-x} - \sqrt{2}}{x - 2}$

Unlike the previous examples, this fraction does not factor. Yet it must simplify, somehow, to eliminate the indeterminate number. Multiply by conjugates to eliminate the radicals from the numerator:

$$\begin{aligned}\lim_{x \to 2} \frac{\sqrt{4-x} - \sqrt{2}}{x - 2} &= \lim_{x \to 2} \frac{(\sqrt{4-x} - \sqrt{2})(\sqrt{4-x} + \sqrt{2})}{(x-2)(\sqrt{4-x} + \sqrt{2})} \\ &= \lim_{x \to 2} \frac{4 - x - 2}{(x-2)(\sqrt{4-x} + \sqrt{2})} \\ &= \lim_{x \to 2} \frac{2 - x}{(x-2)(\sqrt{4-x} + \sqrt{2})}\end{aligned}$$

$$= \lim_{x\to 2} \frac{-1}{\sqrt{4-x}+\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}+\sqrt{2}}$$

$$= \frac{-1}{2\sqrt{2}} \quad \text{or} \quad \frac{-\sqrt{2}}{4}$$

LEARNING OUTCOMES

Evaluate a limit using L'Hopital's Rule.
Determine end behavior using L'Hopital's Rule.

One of the more powerful tools in Calculus for dealing with indeterminate forms and limits is called L'Hopital's Rule.

L'Hopital's Rule:

If $\frac{f(a)}{g(a)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

L'Hopital's Rule allows you to bypass all the algebra in EX 1 and EX 2.

EX 3 Find $\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6}$. (EX 1 again)

$$\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6} \overset{L'H}{=} \lim_{x\to -3} \frac{\frac{d}{dx}\left(2x^3+x^2-13x+6\right)}{\frac{d}{dx}\left(x^2+x-6\right)}$$

$$= \lim_{x\to -3} \frac{6x^2+2x-13}{2x+1}$$
$$= \frac{35}{-5}$$
$$= -7$$

EX 4 Find $\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2}$ using L'Hopital's Rule. (EX 2 again)

Since, at $x=0$, $\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2} = \frac{\sqrt{4-2}-\sqrt{2}}{2-2} = \frac{0}{0}$, the condition for L'Hopital's Rule is satisfied.

$$\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2} \overset{L'H}{=} \lim_{x\to 2} \frac{D_x\left(\sqrt{4-x}-\sqrt{2}\right)}{D_x(x-2)}$$
$$= \lim_{x\to 2} \frac{\frac{1}{2\sqrt{4-x}}(-1)}{1}$$
$$= \frac{-1}{2\sqrt{2}}$$

9-5 Free Response Homework

Evaluate the following limits.

1. $\lim_{x \to -3} \frac{x^2+7x+12}{x^2-9}$

2. $\lim_{x \to -1} \frac{x^2+4x+3}{x^2-3x-4}$

3. $\lim_{x \to 2} \frac{x^2+4x-12}{3x-1}$

4. $\lim_{x \to 2} \frac{4x-12}{x^2-3x-10}$

5. $\lim_{x \to 1} \frac{x^2-x}{e^{x-1}}$

6. $\lim_{x \to 1} \frac{x^3-1}{x^2+3x-4}$

7. $\lim_{x \to 4} \frac{\ln\left(\frac{x}{4}\right)}{\sqrt{x}-2}$

8. $\lim_{x \to -3} \frac{x^2+4x-1}{x^3+3x}$

9. $\lim_{x \to 9} \frac{3-\sqrt{x}}{x^2-81}$

10. $\lim_{x \to 0} \frac{\sqrt{5-x}-\sqrt{5}}{x}$

11. $\lim_{x \to 1} \frac{x^2-1}{\ln x}$

12. $\lim_{x \to -1} \frac{1-x^2}{e^{x+1}-1}$

13. $\lim_{x \to -1} \frac{x+1}{e^{x+1}}$

14. $\lim_{x \to 5} \frac{1-\sqrt{x-5}}{x-6}$

15. $\lim_{x \to -1} \frac{\left(1-x^2\right)^2}{x^3-2x^2-3x}$

16. $\lim_{x \to 2} \frac{x^2+4x-1}{x^3+3x}$